Extremal and Probabilistic Graph Theory
Lecture 19
May 10nd, Tuesday

Lemma 19.1 (Embedding Lemma). *Let $H = (A \cup B, F)$ be a bipartite graph in which $|A| = a$, $|B| = b$, and the vertices in B have degree at most r. If G is a graph with a vertex subset U with $|U| = a$ such that all subsets of U of size r have at least $a + b$ common neighbors, then H is a subgraph of G.*

Definition 19.2. A *topological copy* of a graph H is a graph formed by replacing edges of H by internally vertex disjoint paths. If each of the paths replacing the edges of H has exactly t internal vertices, it is called a *t-subdivision* of H.

Theorem 19.3. *If G is a graph with n vertices and ϵn^2 edges, then G contains a 1-subdivision of a complete graph with $a = \epsilon^{3/2} n^{1/2}$ vertices.*

Proof. $d(G) = 2\epsilon n$. Let $r = 2$, $t = \frac{\log n}{2 \log 1/\epsilon}$ and $m = a + \binom{a}{2}$, $\epsilon \leq 1/2$,

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq \frac{(2\epsilon n)^t}{n^{t-1}} - \frac{n^2}{2}\epsilon^{3t} = 2^t n^{1/2} - \frac{1}{2} n^{1/2} \geq n^{1/2} \geq \epsilon^{3/2} n^{1/2}.$$

Therefore we can apply Lemma 18.1 with these parameters to find a vertex subset U of G with $|U| = a$ such that every pair of vertices in U have at least m common neighbors. Complete the proof using Lemma 19.1. ∎

N. Alon, M. Krivelevich and B. Sudakov show that an n-vertex graph G with ϵn^2 edges contains a 1-subdivision of a complete graph with $\frac{\epsilon}{4} n^{1/2}$ vertices. The power of ϵ cannot be improved.

Definition 19.4. For a graph H, the Ramsey number $r(H)$ is the minimum positive integer N such that every 2-coloring of the edges of K_N contains a monochromatic copy of H.

Definition 19.5. The r-cube Q_r is the r-regular graph with 2^r vertices whose vertex set consists of all binary vectors $\{0,1\}^r$ and two vertices are adjacent if they differ in exactly one coordinate.

Theorem 19.6. $r(Q_r) \leq 2^{3r}$.

Proof. In any 2-edge-coloring of K_N, the denser of the two colors has at least $\frac{1}{2}\binom{N}{2}$ edges. Let $N = 2^{3r}$, let G be the graph of the densest color, so

$$d(G) \geq \frac{2 \times \frac{1}{2}\binom{N}{2}}{N} = N - 1 \geq 2^{-\frac{4}{3}} N.$$

Let $t = \frac{3}{2} r$, $m = 2^r$, and $a = 2^{r-1}$, we have

$$\frac{d^t}{N^{t-1}} - \binom{N}{r}\left(\frac{m}{N}\right)^t \geq 2^{-\frac{4}{3}t} N - N^{r-t}\frac{m^t}{r!} \geq 2^r - 1 \geq 2^{r-1}.$$

Therefore, applying Lemma 18.1 we find a subset U of size 2^{r-1} such that every set of size r in U has at least 2^r common neighbors. Complete the proof by Lemma 19.1. ∎

Let $G = (V, E)$, $R \in V^r$ is a sequence of r vertices of V with replacement $N(R) = \bigcap_{v \in R} N(v)$ and $d(R) = |N(R)|$.

Definition 19.7. R is b-rich if $d(R) \geq b$ and b-poor if $d(R) < b$. A set $Z \subset V$ is (r, b)-rich if every sequence in Z^r is b-rich.

Lemma 19.8 (Two-sided version)**.** *Let $r, s, t \geq 1$ and let G be bipartite graph with average degree d and parts U and V of size n. Suppose $n^{r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4}$. Then there exists $X \subset U$, $Y \subset V$ of size at least $4^{-\frac{1}{s}} d^s n^{1-s}$ such that every X and Y are (r, t)-rich in $G[X, Y]$.*

Proof. Let $\mathcal{F} = \{(S_U, S_V) | S_U \in U^s,\ S_V \in V^s$, and $G[S_U, S_V]$ is a complete bipartite graph.$\}$ Let $N = |\mathcal{F}|$, then

$$\begin{aligned}\sum_{S \in U^s} d(S) &= \#\{(S, v) | S \in U^s,\ v \in V, \text{ and } v \in N(S)\} \\ &= \sum_{v \in V} d(v)^s \geq n \cdot \left(\frac{\sum_{v \in V} d(v)}{n} \right)^s = nd^s,\end{aligned}$$

$$\begin{aligned}N = \sum_{S \in U^s} d(S)^s &\geq n^s \left(\sum \frac{d(S)}{n^s} \right)^s \\ &= n^{s-s^2} \left(\sum_{S \in U^s} d(S) \right)^s = n^{s-s^2} \left(\sum_{v \in V} d(v)^s \right)^s \\ &\geq n^{s-s^2} n^s d^{s^2} = n^{2s-s^2} d^{s^2}.\end{aligned}$$

Uniformly select sequences $(S_U, S_V) \in \mathcal{F}$, let $X = N(S_U)$, $Y = N(S_V)$. A sequence $A \in X^r$ is t-poor if $|N(A \cup S_V)| < t$ and t-rich otherwise. A sequence $A \in X^r$ is t-rich $\Leftrightarrow |N(A \cup S_V)| \geq t \Leftrightarrow |N(A) \cap Y| \geq t$. The expected number of t-poor sequence in X^r is

$$\begin{aligned}&\frac{1}{N} \#\{(S_U, S_V, A) | A \in N(S_U)^r, \text{ and } A \text{ is } t\text{-poor}\} \\ =& \frac{1}{N} \sum_{A \in V^r} \sum_{S' \in V^s} \#\{S : S \in U^s, S \subset N(A \cup S')\} \\ \leq& \frac{1}{N} n^r n^s (t-1)^s \leq n^{s^2-2s} d^{-s^2} n^{r+s} (t-1)^s < \frac{1}{4}.\end{aligned}$$

Therefore the probability that X^r contains a t-poor sequence is less than $\frac{1}{4}$.

A similar statement holds for Y. Then we have with probability $> \frac{1}{2}$ in $G[X, Y]$ X and Y are both (r, t)-rich. (*)

Let $m = 4^{-\frac{1}{s}} d^s n^{1-s}$,then

$$P(|X| < m) = \frac{1}{N} \sum_{S \in V^s, d(S) < m} d(S)^s < \frac{1}{N} n^s m^s \leq n^{s^2-2s} d^{-s^2} n^s \cdot \frac{1}{4} d^{s^2} n^{s-s^2} \leq \frac{1}{4}.$$

The same holds for Y. Then we have probability $> \frac{1}{2}$ both $|X|$ and $|Y|$ are no less than m.

Together with (*) with positive probability X and Y are the required sets. ∎